A/PROF MIN-HSIU HSIEH

41076 METHODS IN QUANTUM COMPUTING

LINEAR ALGEBRA IN DIRAC NOTATION

- Outline
  - Dirac Notation
  - Matrix Functions
  - Tensor Product
  - Trace and Partial Trace

DIRAC NOTATION – BRAKET

$\mathscr{H}$: d-dim Hilbert Space

$$\mathbf{v} = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix}$$

$$= \sum_{i=0}^{d-1} v_i \mathbf{e}_i$$

$$|v\rangle = \sum_{i=0}^{d-1} v_i |i\rangle$$

DIRAC NOTATION - BRAKET

$$|v\rangle = \sum_{i=0}^{d-1} v_i |i\rangle$$

$$\langle v| = |v\rangle^\dagger$$

$$= (v_0^*, \cdots, v_{d-1}^*)$$

DIRAC NOTATION – BRAKET

Inner Product:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u}^\dagger \mathbf{v} \qquad \langle \mathbf{u} | \mathbf{v} \rangle = \sum_{i=0}^{d-1} u_i^* v_i$$

Outer Product:

$$|\mathbf{v}\rangle\langle\mathbf{u}| = \sum_{i=0}^{d-1} u_j^* v_i |i\rangle\langle j|$$

DIRAC NOTATION - LINEAR OPERATOR

$$\mathscr{L}(\mathscr{H}) = \{L : \mathscr{H} \to \mathscr{H}\}$$

$$L = \sum_{i,j} L_{i,j} |i\rangle\langle j|$$

$$L_{i,j} = \langle i | L | j \rangle$$

DIRAC NOTATION – HERMITIAN AND POSITIVE OPERATORS

$H \in \mathscr{L}(\mathscr{H})$ is Hermitian if $H = H^\dagger$

$$H = \sum_i \lambda_i |\nu_i\rangle\langle\nu_i|$$

$P \in \mathscr{L}(\mathscr{H})$ is positive if $\langle v | P | v \rangle \geq 0$

$$\mathscr{L}(\mathscr{H})_+ = \{P : P \geq 0\}$$

MATRIX FUNCTION

For any $f : \mathbb{R} \to \mathbb{R}$, and H Hermitian

$$H = \sum_i \lambda_i |\nu_i\rangle\langle\nu_i|$$

$$f(H) = \sum_i f(\lambda_i) |\nu_i\rangle\langle\nu_i|$$

MATRIX FUNCTION

Convexity:

$$f((1-p)A + pB) \leq (1-p)f(A) + pf(B)$$

Monotone:

$$A \geq B \Rightarrow f(A) \geq f(B)$$

TENSOR PRODUCT

$$|v\rangle_A = \sum_{i=0}^{d_A-1} v_i |i\rangle_A \qquad |u\rangle_B = \sum_{j=0}^{d_B-1} u_j |j\rangle_B$$

$$|v\rangle_A \otimes |u\rangle_B = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} v_i u_j |i\rangle_A \otimes |j\rangle_B$$

TENSOR PRODUCT

$$L = \sum_{i,j=0}^{d_A-1} L_{i,j} |i\rangle\langle j| \qquad M = \sum_{k,\ell=0}^{d_B-1} M_{k,\ell} |k\rangle\langle \ell|$$

$$L \otimes M = \sum_{i,j=0}^{d_A-1} \sum_{k,\ell=0}^{d_B-1} L_{i,j} M_{k,\ell} |i\rangle\langle j|_A \otimes |k\rangle\langle \ell|_B$$

PROPERTY OF TENSOR PRODUCT

1. $(A_1 \otimes \cdots \otimes A_k)(B_1 \otimes \cdots \otimes B_k) = (A_1B_1 \otimes \cdots \otimes A_kB_k)$

2. $(A_1 \otimes \cdots \otimes A_k)^{-1} = (A_1^{-1} \otimes \cdots \otimes A_k^{-1})$

3. $(A_1 \otimes \cdots \otimes A_k)^{\dagger} = (A_1^{\dagger} \otimes \cdots \otimes A_k^{\dagger})$

TRACE

$$\mathrm{Tr} : \mathscr{L}(\mathscr{H}) \to \mathbb{C}$$

$$\mathrm{Tr}\,|j\rangle\langle k| = \langle k|j\rangle = \delta_{j,k}$$

TRACE

$$\mathrm{Tr}\, L = \mathrm{Tr} \sum_{i,j} L_{i,j} |i\rangle\langle j|$$

$$= \sum_{i,j} L_{i,j} \mathrm{Tr}\, |i\rangle\langle j|$$

$$= \sum_{i,j} \langle i|L|j\rangle\langle j|i\rangle$$

$$= \sum_{i} \langle i|L|i\rangle$$

PARTIAL TRACE

$$\mathrm{Tr}_A : \mathscr{L}(\mathscr{H}_{AB}) \to \mathscr{L}(\mathscr{H}_B)$$

$$\mathrm{Tr}_A \, |i\rangle\langle j|_A \otimes |k\rangle\langle \ell|_B = \langle i|j\rangle \, |k\rangle\langle \ell|_B$$

The End of “Linear Algebra in Dirac Notation ”