41076: Methods in Quantum Computing

'Quantum Algorithms' Module

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Abstract

Contents to be covered in this lecture are

1. Deutsch-Jozsa Algorithm
2. Bernstein-Vazirani Algorithm
3. Grover's Algorithm
4. Fourier Transform Algorithm
5. Phase Estimation

The topic of quantum algorithms has attracted significant interests over the past two decades. There have been a substantial amount of results in the area that are worth covering. In fact, advanced lectures on quantum algorithms already exist (see, e.g., [1, 2]).

The 'quantum algorithm' module consists of two lectures. In the first lecture, we will introduce some of the earliest quantum algorithms that could outperform classical algorithms. In particular, we will focus on two algorithms that are building blocks of more sophisticated algorithms, i.e.,

- Quantum Fourier Transform
    1. Phase Estimation
    2. Period Finding
    3. Factoring
    4. Hidden Subgroup
- Grover Search
    1. Quantum Amplitude Amplification
    2. Quantum Minimum Finding
    3. Quantum Counting

1 Deutsch-Jozsa Algorithm

Problem 1 (Deutsch-Jozsa)**.** *For $N = 2^n$, we are given a promise that the bit string $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{Z}_2^N$ satisfies one of the following two conditions:*

1. *(constant) all entries x_i have the same value, or*
2. *(balanced) half the entries are '0' and the other half are '1'.*

The goal is to find out whether $\boldsymbol{x}$ is constant or balanced.

Before going to the quantum algorithm for solving this problem, let us first look at the possible classical solutions. In fact, the classical strategy is very simple. If we want 100% accuracy, in the worse case, one has to *query* at least $\frac{N}{2} + 1$ bits in $\boldsymbol{x}$.

It seemed quite counterintuitive in the beginning that there is a quantum algorithm, proposed by Deutsch and Jozsa, which can produce the correct answer with just a single use of quantum oracle (i.e., quantum unitary). The quantum oracle queries the bit string $\boldsymbol{x}$ only once; hence the Deutsch-Jozsa algorithm has the exponential saving, compared with the classical strategy, in the number of queries to $\boldsymbol{x}$.

Define the quantum oracle

$$O_{\boldsymbol{x}} : |i\rangle \otimes |b\rangle \to |i\rangle \otimes |b \oplus x_i\rangle \tag{1}$$

where $i \in [N]$, $b \in \mathbb{Z}_2$ and $\oplus$ is the binary addition.

Exercise 2. *Show that*

$$O_{\boldsymbol{x}}|i\rangle \otimes |-\rangle = (-1)^{x_i}|i\rangle \otimes |-\rangle, \tag{2}$$

where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

The quantum circuit for the Deutsch-Jozsa algorithm is depicted in Figure 1.

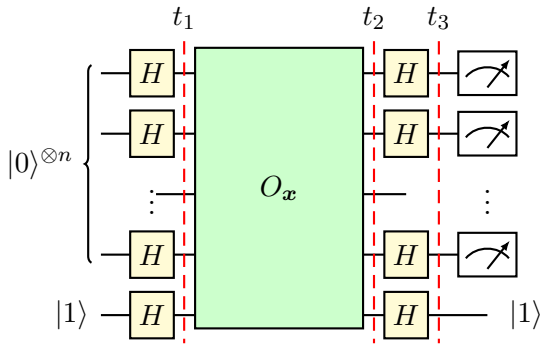


Figure 1: The Deutsch-Jozsa Algorithm.

At the step t_1, the quantum circuit prepares the quantum state

$$|\Psi_{t_1}\rangle = \left(\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle\right) \otimes |-\rangle. \tag{3}$$

At the step t_2, the quantum state becomes

$$|\Psi_{t_2}\rangle = O_{\boldsymbol{x}}|\Psi_{t_1}\rangle \tag{4}$$

$$= \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}(-1)^{x_i}|i\rangle\otimes|-\rangle \tag{5}$$

where the second line employs Eq. (2).

Exercise 3. *Show that*

$$H^{\otimes n}|i\rangle = \frac{1}{\sqrt{N}}\sum_{j}(-1)^{i\cdot j}|j\rangle. \tag{6}$$

At the step t_3, the quantum state is

$$|\Psi_{t_3}\rangle = H^{\otimes(n+1)}|\Psi_{t_2}\rangle \tag{7}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1}(-1)^{x_i}\sum_{j=0}^{N-1}(-1)^{i\cdot j}|j\rangle\otimes|1\rangle. \tag{8}$$

$$= \left(\frac{1}{N}\sum_{i=0}^{N-1}(-1)^{x_i}|0\rangle\otimes|1\rangle\right) + \left(\sum_{i=0}^{N-1}\sum_{j\neq 0}\text{Remaining Terms}\right). \tag{9}$$

Finally, we can compute the probability of measuring the computational basis. In particular, the probability of seeing outcome '0' is

$$\Pr\{\text{Outcome } 0\} = \left|\frac{1}{N}\sum_{i=0}^{N-1}(-1)^{x_i}\right|^2 \tag{10}$$

$$= \begin{cases} 1, \text{if } \boldsymbol{x} \text{ is constant} \\ 0, \text{if } \boldsymbol{x} \text{ is balanced} \end{cases}. \tag{11}$$

Before ending this section, I would like to emphasize that if we allow some small error probability in deciding whether $\boldsymbol{x}$ is constant or balance in the classical setting, the quantum advantage of Deutsch-Jozsa algorithm will disappear completely.

2 Bernstein-Vazirani Problem

Problem 4 (Bernstein-Vazirani)**.** *For $N = 2^n$, we are given $\boldsymbol{x} = (x_1, \cdots, x_N) \in \mathbb{Z}_2^N$ with the promise that there exists some unknown $\boldsymbol{a} \in \mathbb{Z}_2^n$ such that $x_i = (i \cdot \boldsymbol{a}) \bmod 2$. The goal is to decide $\boldsymbol{a}$.*

The Bernstein-Vazirani can be solved in the exact same way as the Deutsch-Jozsa algorithm. Hence, I will leave it as an exercise.

Exercise 5. *Outline the steps for the Bernstein-Vazirani algorithm.*

3 Grover's Search Algorithm

Problem 6 (The search problem)**.** *For $N = 2^n$, we are given a marked item $\boldsymbol{w} \in \mathbb{Z}_2^N$, and the goal is to locate $\boldsymbol{w}$.*

The classical solution is easy to see. In the worse case, the algorithm has to check all N item in order to find $\boldsymbol{x}$.

In the quantum setting, we are given an oracle U_{G} so that

$$U_{\mathrm{G}}|\boldsymbol{x}\rangle = \begin{cases} -|\boldsymbol{x}\rangle, & \text{if } \boldsymbol{x} = \boldsymbol{w} \\ |\boldsymbol{x}\rangle, & \text{otherwise} \end{cases}. \tag{12}$$

In other words, the Grover's oracle can add a phase to the target element $|\boldsymbol{w}\rangle$. We can see that

$$U_{\mathrm{G}} = I - 2|\boldsymbol{w}\rangle\langle\boldsymbol{w}|. \tag{13}$$

We will also define a *diffusion* operator as follows:

$$U_{\mathrm{d}} = 2|\boldsymbol{s}\rangle\langle\boldsymbol{s}| - I, \tag{14}$$

where

$$|\boldsymbol{s}\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle. \tag{15}$$

Note that the diffusion operator can be easily implemented as follows:

$$U_{\mathrm{d}} = H^{\otimes n}\big(2|0\rangle\langle 0|^{\otimes n} - I\big)H^{\otimes n}. \tag{16}$$

The quantum circuit for the Grover's search algorithm is illustrated in Figure 2, where U_{G} and U_{d} are given in Eqs. (13) and (14), respectively. The Grover's algorithm has the following steps.

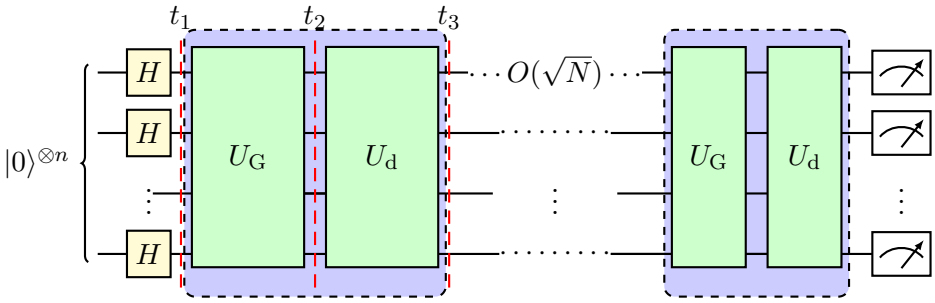


Figure 2: The Grover's Algorithm.

At the step t_1, the algorithm prepares a uniform superposition state:

$$|\Psi_{t_1}\rangle = H^{\otimes n}|0\rangle^{\otimes n} = \left(\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle\right) \equiv |\boldsymbol{s}\rangle. \tag{17}$$

At the step t_2, the protocol employs the Grover's oracle U_{G}:

$$\begin{aligned} |\Psi_{t_2}\rangle &= U_{\mathrm{G}}|\Psi_{t_1}\rangle \\ &= |\boldsymbol{s}\rangle - \frac{2}{\sqrt{N}}|\boldsymbol{w}\rangle, \end{aligned} \tag{18}$$

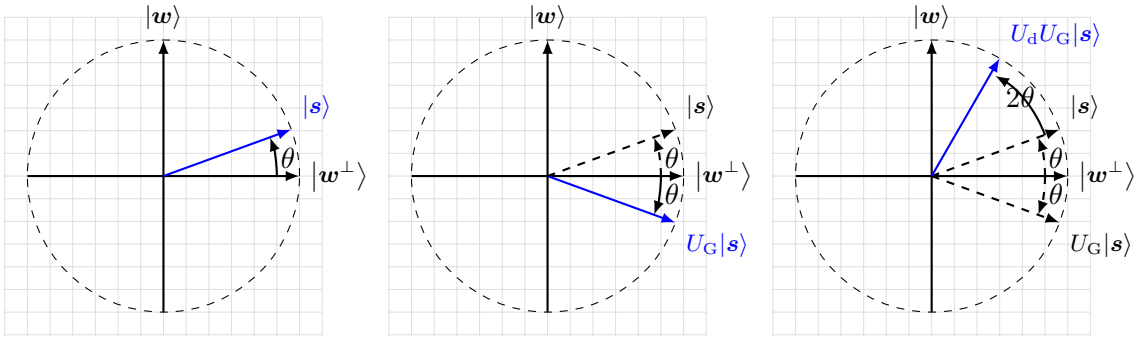
because $\langle \boldsymbol{s}|\boldsymbol{w}\rangle = \frac{1}{\sqrt{N}}$. At the step t_3, the protocol employs the diffusion oracle U_{d}:

$$|\Psi_{t_3}\rangle = U_{\mathrm{d}}|\Psi_{t_2}\rangle \tag{19}$$

$$= (2|\boldsymbol{s}\rangle\langle \boldsymbol{s}| - I)\left(|\boldsymbol{s}\rangle - \frac{2}{\sqrt{N}}|\boldsymbol{w}\rangle\right) \tag{20}$$

$$= \left(\frac{N-4}{N}|\boldsymbol{s}\rangle\right) + \frac{2}{\sqrt{N}}|\boldsymbol{w}\rangle. \tag{21}$$

We say that "one iteration" of the Grover's search algorithm consists of the employment of U_{G} followed by U_{d}. In the following, we aim to prove show that (i) the protocol can find the target $\boldsymbol{w}$ with $O(\sqrt{N})$ iterations with successful probability ≈ 1 and (ii) this is optimal given access to quantum computers.



(a) Step t_1 **(b)** Step t_2 **(c)** Step t_3

Figure 3: Geometric illustration of Grover's search procedure

The easiest way to prove statement (i) is by the following geometric argument in Figure 3. Denote

$$\left|\boldsymbol{w}^\perp\right\rangle = \frac{1}{\sqrt{N-1}}\sum_{\boldsymbol{x}\neq\boldsymbol{w}}|\boldsymbol{x}\rangle.$$

One can see that the uniform superposition state $|\boldsymbol{s}\rangle$ at the step t_1 can be decomposed into

$$|\boldsymbol{s}\rangle = \sqrt{\frac{1}{N}}|\boldsymbol{w}\rangle + \sqrt{\frac{N-1}{N}}\left|\boldsymbol{w}^\perp\right\rangle, \tag{22}$$

and the angle θ in Figure 3a corresponds to

$$\sin\theta = \sqrt{\frac{1}{N}}, \qquad \cos\theta = \sqrt{\frac{N-1}{N}}. \tag{23}$$

At the second step t_2, application of $U_{\rm G}$ leads to

$$\begin{aligned} U_{\rm G}|\boldsymbol{s}\rangle &= -\sqrt{\frac{1}{N}}|\boldsymbol{w}\rangle + \sqrt{\frac{N-1}{N}}\Big|\boldsymbol{w}^{\perp}\Big\rangle \\ &= -\sin\theta|\boldsymbol{w}\rangle + \cos\theta\Big|\boldsymbol{w}^{\perp}\Big\rangle. \end{aligned} \tag{24}$$

Geometrically, the oracle $U_{\rm G}$ reflects the vector $|\boldsymbol{s}\rangle$ along the axis $\big|\boldsymbol{w}^{\perp}\big\rangle$ in Figure 3b. Finally, application of $U_{\rm d}$ at the third step t_3 to Eq. (24) is equivalent to reflect the state $U_{\rm G}|\boldsymbol{s}\rangle$ along the axis $|\boldsymbol{s}\rangle$. Therefore, the application of $U_{\rm d}U_{\rm G}$ yields

$$U_{\rm d}U_{\rm G}|\boldsymbol{s}\rangle = \sin 3\theta|\boldsymbol{w}\rangle + \cos 3\theta\Big|\boldsymbol{w}^{\perp}\Big\rangle. \tag{25}$$

By induction, after k iterations, we have

$$(U_{\rm d}U_{\rm G})^k|\boldsymbol{s}\rangle = \sin(2k+1)\theta|\boldsymbol{w}\rangle + \cos(2k+1)\theta\Big|\boldsymbol{w}^{\perp}\Big\rangle. \tag{26}$$

If we measure after k iterations, the probability of obtaining the target element $\boldsymbol{w}$ is

$$p_k := \Pr\{\boldsymbol{w} \text{ appears}\} = \sin((2k+1)\theta)^2. \tag{27}$$

If we choose $k = \frac{\pi}{4\theta} - \frac{1}{2}$, then the Grover's algorithm will produce the state $|\boldsymbol{w}\rangle$ with certainty because $p_k = 1$. However $k = \frac{\pi}{4\theta} - \frac{1}{2}$ will unlikely be an integer, but we can still show that if $\tilde{k}$ is an integer closest to k and $1 \ll N$, then the failure probability decays proportional to N:

$$\begin{aligned} 1 - p_{\tilde{k}} &= \cos((2\tilde{k}+1)\theta)^2 \\ &= \cos((2k+1)\theta + 2(\tilde{k}-k)\theta)^2 \\ &= \cos(\pi/2 + 2(\tilde{k}-k)\theta)^2 \\ &= \sin(2(\tilde{k}-k)\theta)^2 \\ &\le \sin(\theta)^2 \\ &= \frac{1}{N}, \end{aligned} \tag{28}$$

where the first inequality follows because $|k - \tilde{k}| \le 1/2$. Since $\arcsin\theta \ge \theta$, then

$$\tilde{k} \le \frac{\pi}{4\theta} = \frac{\pi}{4}\sqrt{N}. \tag{29}$$

Hence, we can see that the number of iteration is $O(\sqrt{N})$.

Exercise 7. *Prove that quantum search algorithms with number of queries less than $O(\sqrt{N})$ can not be optimal.*

Exercise 8 (Quantum amplitude amplification algorithm)**.** *Let*

$$|\psi\rangle = \sin\theta|\psi_{=}\rangle + \cos\theta|\psi_{\perp}\rangle,$$

where $\langle\psi_{=}|\psi_{\perp}\rangle = 0$, and let the projector $P|\psi_{=}\rangle = |\psi_{=}\rangle$ Let $U_{\rm A} = U_{\psi}U_P$, where

$$U_{\psi} = I - 2|\psi\rangle\langle\psi| \tag{30}$$

$$U_P = 2P - I. \tag{31}$$

Show that

$$(U_{\mathrm{A}})^k|\psi\rangle = \sin((2k+1)\theta)|\psi_=\rangle + \cos((2k+1)\theta)|\psi_\perp\rangle,$$

and the largest amplitude on $|\psi_=\rangle$ can be achieved with $k \approx O(\sqrt{N})$. In other words, the oracle U_{A} can amplify the amplitude of the constituent state $|\psi_=\rangle$ with suitably chosen k.

Exercise 9 (Quantum minimum finding algorithm, [3])**.** *Given an unordered list $\{f(i)\}_{i=1}^N$ with N items, the goal of the minimum finding algorithm is to find an index k^*, i.e.,*

$$k^* = \arg\min_{i\in[N]} f(i) \ . \tag{32}$$

Show that there exists a quantum algorithm that finds the index k^* defined in Eqn. (32) with probability at least $1/2$ with $O(\sqrt{N})$ queries.

4 Quantum Fourier Transform

The discrete Fourier transform of a set $\{x_0, \cdots, x_{N-1}\}$ of N elements is defined as

$$X_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{\frac{i2\pi}{N}jk}. \tag{33}$$

Exercise 10. *Let X_k be Fourier transform of $\{x_k\}_{k\in[N]}$. Then*

$$\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X_k|^2. \tag{34}$$

Exercise 11. *For $N = 2^n$, the runtime to compute X_k is $O(N \log N)$.*

In the following, we will denote

$$\omega_N := e^{\frac{i2\pi}{N}} \tag{35}$$

the N-th root of unity. Let U_{F} be the square matrix whose (i,j)-th element is $\frac{1}{\sqrt{N}}\omega_N^{ij}$.

Exercise 12. *Show that U_{F} is unitary.*

The definition of Fourier transform in Eq. (33) can be extended to the quantum setting

$$|\Psi_k\rangle := U_{\mathrm{F}}|k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{i2\pi}{N}jk}|j\rangle. \tag{36}$$

It is crucial to note that the state $|\Psi_k\rangle$ is a product state when $N = 2^n$. Therefore there exists an efficient quantum circuit to prepare this state. To show that, consider the binary representation of $k \equiv (k_1, \cdots, k_n) \in [N]$, where k_1 is the most significant bit, i.e.,

$$k = k_1 2^{n-1} + k_2 2^{n-2} + \cdots + k_n 2^0, \tag{37}$$

and we write

$$k/2^n = 0.k_1k_2\cdots k_n = \sum_{\ell=1}^{n} k_\ell 2^{-\ell}.$$

Take for example, $k = 5 = (1, 0, 1)$ and $n = 3$, therefore $5/8 = 0.101$. Thus

$$\begin{aligned}
|\Psi_k\rangle &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{i2\pi jk/2^n} |j\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{j_1=0}^{1} \cdots \sum_{j_n=0}^{1} e^{i2\pi \left(\sum_{\ell=1}^{n} j_\ell/2^{-\ell}\right)k} |j_1, j_2, \cdots, j_n\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{j_1=0}^{1} \cdots \sum_{j_n=0}^{1} \bigotimes_{\ell=1}^{n} e^{i2\pi j_\ell k/2^{-\ell}} |j_\ell\rangle \\
&= \frac{1}{\sqrt{2^n}} \bigotimes_{\ell=1}^{n} \left[\sum_{j_\ell=0}^{1} e^{i2\pi j_\ell k/2^{-\ell}} |j_\ell\rangle \right] \\
&= \bigotimes_{\ell=1}^{n} \frac{1}{\sqrt{2}} \Big(|0\rangle + e^{i2\pi k/2^\ell} |1\rangle \Big), \\
&:= \bigotimes_{\ell=1}^{n} |\Phi_\ell\rangle, \qquad (38)
\end{aligned}$$

where in the last line we denote

$$|\Phi_\ell\rangle := \frac{1}{\sqrt{2}} \Big(|0\rangle + e^{i2\pi k/2^\ell} |1\rangle \Big), \qquad (39)$$

and $k/2^\ell = 0.k_{n-\ell-1} \cdots k_n$, the ℓ least significant bits of k because the first $n - \ell$ most significant bits of k have no effect on the value ($e^{i2\pi m} = 1$ for $m \in \mathbb{N}$).

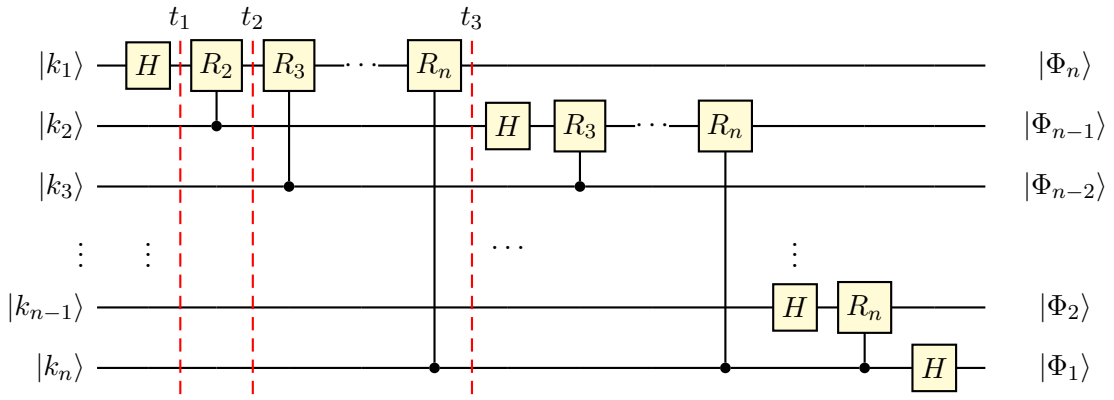


Figure 4: Circuit for quantum Fourier transform.

The implementation of Eq. (38) is given in Figure 4, where

$$R_m = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^m} \end{pmatrix}. \qquad (40)$$

At the step t_1, the quantum state is transformed into

$$|k_1\rangle \otimes \cdots \otimes |k_n\rangle \to \frac{1}{\sqrt{2}} \Big(|0\rangle + e^{i2\pi 0.k_1} |1\rangle \Big) \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle, \qquad (41)$$

because

$$e^{i2\pi 0.k_1} = e^{i2\pi\frac{k_1}{2}} = \begin{cases} -1 & \text{when } k_1 = 1 \\ 1 & \text{when } k_1 = 0 \end{cases}. \tag{42}$$

At the step t_2, we have

$$\frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i2\pi 0.k_1k_2}|1\rangle\Big) \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle, \tag{43}$$

because when $k_2 = 0$, the state in Eq. (43) is the same as Eq. (41), and when $k_2 = 1$, a phase of $e^{i2\pi/2^2}$ is applied. Following the same derivation, the state at the step t_3 will become

$$\frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i2\pi 0.k_1k_2\cdots k_n}|1\rangle\Big) \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle. \tag{44}$$

We can see that the number of gates in Figure 4 is

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}, \tag{45}$$

that is exponentially less than the classical fast Fourier transform which requires $O(n2^n)$ gates (Exercise 11).

5 Phase Estimation

In this section, we will introduce the quantum phase estimation protocol. The crucial component of the quantum phase estimation protocol is the quantum Fourier transform.

Problem 13. *Given a unitary U and its eigenvector $|\nu\rangle$, estimate the corresponding eigenvalue $\lambda = e^{i2\pi\varphi}$.*

Theorem 14. *The quantum phase estimation algorithm, illustrated in Figure 5, can estimate the value of φ to the additive error ε with high probability, using $O(\log(\frac{1}{\varepsilon}))$ qubits and $O(\frac{1}{\varepsilon})$ controlled-U operations.*

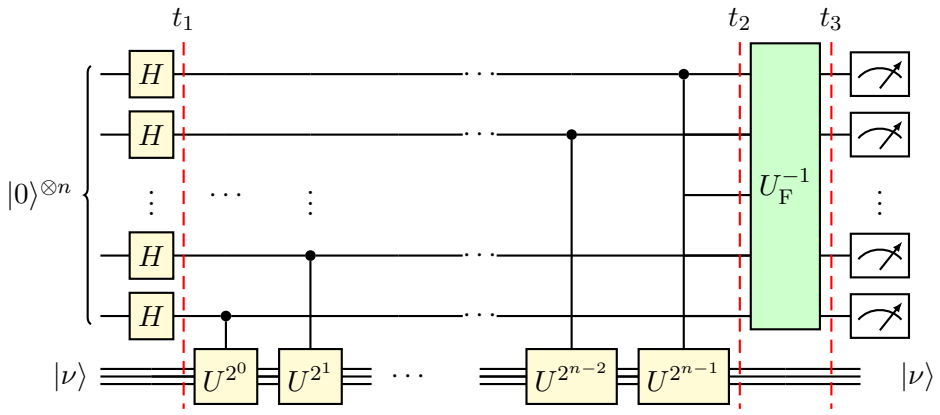


Figure 5: Quantum Phase Estimation.

Proof. Let $N = 2^n$. At the step t_1, the overall state is

$$|\Psi_{t_1}\rangle = \left(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle\right)\otimes|\nu\rangle. \tag{46}$$

Since $U|\nu\rangle = e^{i2\pi\varphi}|\nu\rangle$, we have

$$U^{2^j}|\nu\rangle = e^{i2\pi 2^j\varphi}|\nu\rangle. \tag{47}$$

Then the overall state at the step t_2 is

$$|\Psi_{t_2}\rangle = \left(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}e^{i2\pi\varphi k}|k\rangle\right)\otimes|\nu\rangle. \tag{48}$$

At the step t_3, the overall state is

$$|\Psi_{t_3}\rangle = (U_{\mathrm{F}}^{-1}\otimes I)|\Psi_{t_2}\rangle = \left(\frac{1}{N}\sum_{y=0}^{N-1}\sum_{k=0}^{N-1}e^{\frac{i2\pi k}{N}(N\varphi-y)}|y\rangle\right)\otimes|\nu\rangle. \tag{49}$$

Finally, the probability of obtaining the outcome $a \in [N]$ is

$$\begin{aligned}
\Pr\{\text{Outcome } a\} &= \left|\langle a|\frac{1}{N}\sum_{y=0}^{N-1}\sum_{k=0}^{N-1}e^{\frac{i2\pi k}{N}(N\varphi-y)}|y\rangle\right|^2 &(50)\\
&= \left|\frac{1}{N}\sum_{k=0}^{N-1}e^{\frac{i2\pi k}{N}(N\varphi-a)}\right|^2 &(51)\\
&= \begin{cases}1 & N\varphi = a\\ \left|\frac{1}{N}\sum_{k=0}^{N-1}e^{\frac{i2\pi k}{N}\delta}\right|^2 & N\varphi - a = \delta \neq 0\end{cases}. &(52)
\end{aligned}$$

For the ideal case where $N\varphi = a$ is an integer, the estimation is exact. However, when $N\varphi - a = \delta \neq 0$, we can show that

$$\begin{aligned}
\Pr\{\text{Outcome } a\} &= \left|\frac{1}{N}\sum_{k=0}^{N-1}e^{\frac{i2\pi k}{N}\delta}\right|^2 &(53)\\
&= \frac{1}{N^2}\left|\frac{1-e^{i2\pi\delta}}{1-e^{\frac{i2\pi}{N}\delta}}\right|^2 &(54)\\
&= \frac{1}{N^2}\frac{|\sin\pi\delta|^2}{|\sin\frac{\pi\delta}{N}|^2} &(55)\\
&\geq \frac{|\sin\pi\delta|^2}{|\pi\delta|^2} &(56)\\
&\geq \frac{|2\delta|^2}{|\pi\delta|^2} &(57)\\
&= \frac{4}{\pi^2} &(58)
\end{aligned}$$

where the third line uses

$$|1 - e^{i2x}|^2 = 4|\sin x|^2,$$

the fourth line uses $|\sin \frac{\pi\delta}{N}| \leq |\frac{\pi\delta}{N}|$ because $|\delta| < 1/2$, and the fifth line uses $|\sin \pi\delta| \geq |2\delta|$ because $|\delta| < 1/2$.

□

References

[1] Andrew M. Childs, *Lecture notes on quantum algorithms*, 2017.

[2] Ronald de Wolf, *Quantum computing: Lecture notes*, 2019.

[3] Christoph Durr and Peter Hoyer, *A quantum algorithm for finding the minimum*, 1996.